

DEGENERATE AND STABLE YANG-MILLS-HIGGS PAIRS

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ABSTRACT. In this paper, we introduce some notions on the pair consisting of a Chern connection and a Higgs field closely related to the first and second variation of Yang-Mills-Higgs functional, such as strong Yang-Mills-Higgs pair, degenerate Yang-Mills-Higgs pair, stable Yang-Mills-Higgs pair. We investigate some properties of such pairs.

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1. INTRODUCTION

Since 1950s, Yang-Mills theory first explored by several physicists had a profound impact on the developments of differential and algebraic geometry. A remarkable fruit owed to Donaldson is constructing invariants of 4-manifolds via studying the homology of the moduli space of anti-self-dual $SU(2)$ -connections, where technical challenges come from Uhlenbeck compactification of moduli space and handling singularities through the metric perturbations[1, 2]. In 1987 Hitchin considered the 2-dimensional reduction of the self-dual Yang-Mills equations on \mathbb{R}^4 as a manner of symmetry breaking, then he introduced a $(1,0)$ -form valued in adjoint vector bundle ϕ called the Higgs field for the Riemann surface, which is described by the so-called Hitchin self-duality equations[3]:

$$\begin{aligned} F_A + [\phi, \bar{\phi}] &= 0, \\ d_A'' \phi &= 0. \end{aligned}$$

Influenced by the Hitchin's work, Simpson generalized the conception of Higgs field to the higher dimensional case[4], and he made great innovations in various areas of algebraic geometry[5, 6, 7]. Since then Higgs bundles have emerged in the last two decades as a central object of study in geometry, with several links to physics and number theory.

Let us first recall some basic definitions.

Definition 1.1. ([8, 9, 10]) Let X be an n -dimensional compact Kähler manifold with Kähler form ω , and let Ω_X^1 be the sheaf of holomorphic 1-forms on X . A Higgs sheaf over X is a coherent sheaf E of dimension n over X , together with a morphism $\phi : E \rightarrow E \otimes \Omega_X^1$ of \mathcal{O}_X -modules (that is usually called the Higgs field), such that the morphism $\phi \wedge \phi : E \rightarrow E \otimes \Omega_X^2$ vanishes. A Higgs bundle is a locally-free Higgs sheaf. A subsheaf F of E is called the Higgs subsheaf if $\phi(F) \subset F \otimes \Omega_X^1$, i.e. the pair

$F = (F, \phi|_F)$ becomes itself a Higgs sheaf. Let (E_1, ϕ_1) and (E_2, ϕ_2) be two Higgs sheaves over X . A morphism between them is a map $E_1 \rightarrow E_2$ such that the following diagram commutes

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_1 \otimes \Omega_X^1 \\ f \downarrow & & f \otimes 1 \downarrow \\ E_2 & \xrightarrow{\phi_2} & E_2 \otimes \Omega_X^1. \end{array}$$

A Higgs sheaf (E, ϕ) over X is called ω -stable (resp. ω -semistable) if it is torsion-free, and for any Higgs subsheaf F with $0 < \text{rank}(F) < \text{rank}(E)$ the inequality $\mu_\omega(F) < \mu_\omega(E)$ (resp. \leq) holds, where the slope $\mu_\omega(E)$ of E is defined by $\mu_\omega(E) := \frac{\deg_\omega(E)}{\text{rank}(E)} = \frac{\int_X c_1(E) \wedge \omega^{n-1}}{\text{rank}(E) \int_X \omega^n}$. We say that a ω -semistable Higgs sheaf is ω -polystable if it decomposes into a direct sum of ω -stable Higgs sheaves.

Here we briefly mention some significant results of Simpson[5, 6, 7] which will be used in this paper. Let (E, ϕ) be a Higgs bundle over a compact Kähler manifold (X, ω) , then it is ω -polystable if and only if there exists a Hermitian-Yang-Mills-Higgs metric on it. This result as an extension of the Hitchin-Kobayashi correspondence for Higgs bundles is even true for manifolds which are not necessarily compact (satisfying some additional analytic requirements). In particular, there is an equivalence of categories between the category of semisimple flat bundles and the category of polystable Higgs bundles with vanishing (first and second) Chern classes, both being equivalent to the category of harmonic bundles. If X is a smooth projective variety, there is a moduli space \mathcal{M} as a quasi-projective variety of polystable Higgs bundles with vanishing Chern class and there is a natural action of \mathbb{C}^* on \mathcal{M} via multiplying the Higgs field by a non-zero complex number c . Simpson showed that the limit as c goes to zero exists and is unique. The limit is therefore a fixed point of the \mathbb{C}^* -action, and is made into the variation of polarized Hodge structures.

A Yang-Mills-Higgs system is a collection of data $\{(E, \phi), h, d_A\}$ for a Higgs bundle (E, ϕ) with a Hermitian metric h and the corresponding Chern connection d_A on E , where the pair (A, ϕ) is called a Hitchin pair. These data give rise to a non-metric connection $\mathcal{D}_{(A, \phi)}$ called the Hitchin-Simpson connection on E in the following way: $\mathcal{D}_{(A, \phi)} = d_A + \phi + \phi^*$ where $\phi^* : E \rightarrow E \otimes \Omega_X^1$ is the adjoint of the Higgs field with respect to the Hermitian metric h , namely $h(\theta(Y)s, t) = h(s, \theta^*(Y)t)$ for the complex tangent vector Y and the sections s, t of E . The curvature $\mathcal{R}_{(A, \phi)}$ of the Hitchin-Simpson connection is given by $\mathcal{R}_{(A, \phi)} = \mathcal{D}_{(A, \phi)}^2 = (d_A + \phi + \phi^*) \wedge (d_A + \phi + \phi^*) = F_A + [\phi, \phi^*] + d'_A(\phi) + d''_A(\phi^*)$, where $F_A = d_A^2$ denotes the curvature of the Chern connection d_A decomposed into $(1,0)$ -part d'_A and $(0,1)$ -part d''_A . Let $L : \Lambda^{p,q} \rightarrow \Lambda^{p+1,q+1}$ be the operator of the multiplication by the Kähler form ω , Λ be the adjoint of L , and denote the $(1,0)$ -part and $(0,1)$ -part of the Hitchin-Simpson connection $\mathcal{D}_{(A, \phi)}$ by $\mathcal{D}'_{(A, \phi)}$ and $\mathcal{D}''_{(A, \phi)}$ respectively. The following Kähler identities can be easily checked:

$$\begin{aligned} i[\Lambda, \mathcal{D}'_{(A, \phi)}] &= -(\mathcal{D}''_{(A, \phi)})^*, \\ i[\Lambda, \mathcal{D}''_{(A, \phi)}] &= (\mathcal{D}'_{(A, \phi)})^*. \end{aligned} \tag{1.1}$$

Let $\Theta_{(A, \phi)}$ stand for the anti-Hermitian part of $\mathcal{D}_{(A, \phi)}$, which is exactly the $(1,1)$ -part $F_A + [\phi, \phi^*]$. We define the mean curvature $\mathcal{K}_{(A, \phi)}$ of the Hitchin-Simpson connection, just by contraction of its curvature with the operator $i\Lambda$, i.e. $\mathcal{K}_{(A, \phi)} = i\Lambda \mathcal{R}_{(A, \phi)} = i\Lambda \Theta_{(A, \phi)} \in \text{End}(E)$ or in terms of local frame field $\{e_i\}_{i=1}^r$ of E and local coordinates $\{z_\alpha\}_{\alpha=1}^n$ of X , $\mathcal{K}_j^i = \omega^{\alpha\bar{\beta}} \mathcal{R}_{j\alpha\bar{\beta}}^i$.

For a Yang-Mills-Higgs system, one may attempt to solve the following nonlinear equation

$$\det(\mathcal{K}_{(A, \phi)}) = \lambda \tag{1.2}$$

for a constant λ . If the solution of the equation above exists the corresponding system is called a special Yang-Mills-Higgs system, in particular, it is called a degenerate Yang-Mills-Higgs system for the case $\lambda = 0$. When h is a weak Hermitian-Yang-Mills metric on a Higgs bundle (E, ϕ) , that is it satisfies the equation $\mathcal{K}_{(A, \phi)} = f \cdot \text{Id}_E$ for a real function f defined on X , we can obtain a special Yang-Mills-Higgs system $\{(E, \phi), \tilde{h}, d_A\}$ by taking \tilde{h} to be a conformal transformation of h such that \tilde{h} is a Hermitian-Yang-Mills metric. Indeed, since there is a solution u for the equation $\Delta u = c - f$ where c is a constant determined by $c \int_X \omega^n = \int_X f \omega^n$, we only need to let $\tilde{h} = e^u h$ which is the desired Hermitian-Yang-Mills metric with

the constant factor c . More generally, a Yang-Mills-Higgs pair (A, ϕ) (see the following definition) may produce a special Yang-Mills-Higgs system. As a special case, this pair can be realized as the critical point of Yang-Mills-Higgs functional, called the strong Yang-Mills-Higgs pair (see the following definition). Our Yang-Mills-Higgs functional is a natural generation of Yang-Mills functional by replacing the curvature of Chern connection with that of Hitchin-Simpson connection. By calculate the first variation of functional, one can introduce the following flow equations for a pair (A, ϕ)

$$\begin{aligned}\frac{\partial A}{\partial t} &= (d_A'' - d_A')(\mathcal{K}_{(A, \phi)}), \\ \frac{\partial \phi}{\partial t} &= [\phi, \mathcal{K}_{(A, \phi)}].\end{aligned}\tag{1.3}$$

It is easy to check that the Hitchin pair will be preserved by this flow.

Now we introduce some concepts running through this paper.

Definition 1.2. Let $\{(E, \phi), h, d_A\}$ be a Yang-Mills-Higgs system over an n -dimensional compact Kähler manifold (X, ω) . The associated Hitchin pair (A, ϕ) is called

- a Yang-Mills-Higgs pair if it satisfies the following equation

$$d_A'(\Lambda(F_A + [\phi, \phi^*])) = [\phi, \Lambda(F_A + [\phi, \phi^*])].\tag{1.4}$$

- a degenerate Yang-Mills-Higgs pair if it is a Yang-Mills-Higgs pair with the property that $\det[\Lambda(F_A + [\phi, \phi^*])] = 0$ at some point of X .
- a strong Yang-Mills-Higgs pair if it is subject to the equations

$$\begin{aligned}d_A'(\Lambda(F_A + [\phi, \phi^*])) &= d_A''(\Lambda(F_A + [\phi, \phi^*])) = 0, \\ [\Lambda(F_A + [\phi, \phi^*]), \phi] &= [\Lambda(F_A + [\phi, \phi^*]), \phi^*] = 0.\end{aligned}\tag{1.5}$$

- a Hermitian-Yang-Mills-Higgs pair if it satisfies the equation

$$i\Lambda(F_A + [\phi, \phi^*]) = \lambda Id_E\tag{1.6}$$

for the constant $\lambda = \frac{2\pi n}{\int_X \omega^n} \mu_\omega(E)$.

Obviously a Hermitian-Yang-Mills-Higgs pair is a strong Yang-Mills-Higgs pair, and a Hermitian-Yang-Mills-Higgs pair is a degenerate Yang-Mills-Higgs pair if and only if $\deg_\omega(E) = 0$. Existence of such pairs is a strong constraint on the Yang-Mills-Higgs system. In section 2, we will exhibit the mutual restriction of these constraints and stability conditions in algebraic geometry. For example, strongness and degeneracy conditions together generally force the Higgs bundle to split, then by the principle of curvature decreases in Higgs subbundles, one can show that if the associated Hitchin pair is not a Hermitian-Yang-Mills-Higgs pair, the Higgs bundle in the Yang-Mills-Higgs system cannot be semistable.

Deformation of Yang-Mills-Higgs system is described by the deformation of the pair (d_A'', ϕ) which is controlled by a series of equations analogous to the Maurer-Cartan equation in the deformation theory of holomorphic vector bundle. The obstruction of deformation is also characterised by certain second order cohomology. Via calculating the second variation of Yang-Mills-Higgs functional, one can fix which admitted deformation is stable for a given strong Yang-Mills-Higgs pair or if a strong Yang-Mills-Higgs pair is stable with respect to a chosen deformation. Due to the present of Higgs fields, there is no strong Yang-Mills-Higgs pair that is stable along arbitrary admitted deformation (e.g. \mathbb{C}^* -action on Higgs fields). Such stability condition in the sense of differential geometry is reduced to judge the positive-definiteness of a Hermitian quadratic form.

The remainder of this paper is organized as follows. We first introduce the Yang-Mills-Higgs functional for the higher dimensional case as a analog of that for Riemann surface. Next we study some properties of the pairs associated with a Yang-Mills-Higgs system, for example, the interaction with the stability of Higgs bundle and Higgs cohomology. In the last section, we consider the deformation of Hitchin pair and establish the notion of stability of strong Yang-Mills-Higgs pair along the admitted deformation via calculating the second variation of Yang-Mills-Higgs functional.

2. PAIRS ASSOCIATED WITH THE YANG-MILLS-HIGGS SYSTEM

2.1. Yang-Mills-Higgs Functional. Fix a Higgs bundle (E, ϕ) over a compact Riemann surface Σ , then the space $\mathcal{M}(E)$ of holomorphic structures on E as a smooth complex Hermitian vector bundle can be identified with an affine space locally modeled on $\Omega^{0,1}(\text{End}(E))$, i.e. $\mathcal{M}(E) = d_A'' + \mathcal{A}^{0,1}(\text{End}(E))$ for a fixed operator d_A'' , and the Higgs field ϕ takes value in $H^0(\Sigma, K \otimes \text{End}(E)) \simeq H^1(\Sigma, \text{End}(E))^\vee$ where K denotes the canonical line bundle on Σ . The tangent space to $T^*\mathcal{M}(E)$ at any point can be naturally identified with the direct sum $\mathcal{A}^{0,1}(\text{End}(E)) \oplus \mathcal{A}^{1,0}(\text{End}(E))$. Under this identification the metric on $T^*\mathcal{M}(E)$ is given by

$$g((\psi_1, \phi_1), (\psi_2, \phi_2)) = i \left(\int_{\Sigma} \psi_1^* \hat{\wedge} \psi_2 + \phi_1 \hat{\wedge} \phi_2^* + \int_{\Sigma} \psi_2^* \hat{\wedge} \psi_1 + \phi_2 \hat{\wedge} \phi_1^* \right)$$

where $(\psi_i, \phi_j) \in \mathcal{A}^{0,1}(\text{End}(E)) \oplus \mathcal{A}^{1,0}(\text{End}(E))$ and for $\psi_1 = f \otimes u, \psi_2 = g \otimes v$ with $f, g \in \mathcal{A}^{0,1}(\Sigma), u, v \in \text{End}(E)$, $\psi_1^* \hat{\wedge} \psi_2 = \sum_i h(v(e_i), u(e_i)) \bar{f} \wedge g$. Moreover there are compatible complex structures I, J, K defined by

$$\begin{aligned} I(\psi, \phi) &= (i\psi, i\phi), \\ J(\psi, \phi) &= (i\phi^*, -i\psi^*), \\ K(\psi, \phi) &= (-\phi^*, \psi^*). \end{aligned}$$

satisfying the usual quaternionic relations, namely $I^2 = J^2 = K^2 = IJK = -1$. This defines the hyperkähler structure on $T^*\mathcal{M}(E)$. Let \mathcal{G} denote the gauge group of E , which acts on $T^*\mathcal{M}(E)$ preserving the hyperkähler structure. The moment maps for this action are given by[3]

$$\begin{aligned} \mu_I &= F_A + [\phi, \phi^*], \\ \mu_J &= -i(d_A''\phi + d_A'\phi^*), \\ \mu_K &= -d_A''\phi + d_A'\phi^*. \end{aligned}$$

In analogy with the Yang-Mills functional, the full Yang-Mills-Higgs functional is defined to be the norm-square of the hyperkähler moment map, that is we specify

$$YMH(A, \phi) = \int_{\Sigma} (|F_A + [\phi, \phi^*]|^2 + 4|d_A''\phi|^2) dV. \quad (2.1)$$

We can generalize it to the higher dimensional manifold X , thus we consider the following functional

$$YMH(A, \phi) = \int_X |F_A + [\phi, \phi^*] + d_A(\phi + \phi^*)|^2 dV. \quad (2.2)$$

Proposition 2.1. *Let (E, ϕ) be a polystable Higgs bundle on the Kähler manifold (X, ω) , then we have the inequality*

$$||F_A||^2 + 3||[\phi, \phi^*]||^2 + 2||\nabla_A \phi||^2 \geq 2\langle \mathbb{R} \lrcorner \phi, \phi \rangle, \quad (2.3)$$

where the L^2 -norm $|| \cdot ||$ and the global L^2 -inner product $\langle \cdot, \cdot \rangle$ are with respect to the Hermitian-Kähler metric g_ω associated with Kähler form ω on X and the Hermitian-Yang-Mills-Higgs metric h on E , and A is the Chern connection with respect to h , \mathbb{R} denotes the Ricci curvature tensor of X .

Proof. Generally we consider the following functional without the requirement that ϕ is a Higgs field

$$\begin{aligned} \mathcal{F}(A, \phi) &= ||F_A + [\phi, \phi^*]||^2 + 4||d_A''\phi||^2 \\ &= ||F_A||^2 + ||[\phi, \phi^*]||^2 + 2\langle F_A, [\phi, \phi^*] \rangle. \end{aligned}$$

The term coupling the curvature and the Higgs field can be calculated as

$$\begin{aligned}
2\langle F_A, [\phi, \phi^*] \rangle &= 2 \int_X \sum_i g_\omega \otimes h(F_A(e_i), [\phi, \phi^*](e_i)) = 2 \int_X \sum_i h((F_A)_{\mu\bar{\nu}}(e_i), [\phi^{\bar{\mu}}, (\phi^*)^\nu](e_i)) \\
&= \int_X \sum_i h((F_A)_{\mu\bar{\nu}}\phi^{\bar{\nu}}(e_i), \phi^{\bar{\mu}}(e_i)) - h(\phi^{\bar{\nu}}(F_A)_{\mu\bar{\nu}}(e_i), \phi^{\bar{\mu}}(e_i)) \\
&\quad + \int_X h(\phi^{\bar{\mu}}(e_i), (F_A)_{\mu\bar{\nu}}\phi^{\bar{\nu}}(e_i)) - h(\phi^{\bar{\mu}}(e_i), \phi^{\bar{\nu}}(F_A)_{\mu\bar{\nu}}(e_i)) \\
&= 2\text{Re}\langle F_A \lrcorner \phi, \phi \rangle,
\end{aligned}$$

where $\phi^{\bar{\nu}} = g_{\omega}^{\mu\bar{\nu}}\phi_\mu$, $(\phi^*)^\nu = g_{\omega}^{\nu\bar{\mu}}\phi_\mu^*$. On the other hand, by the Bochner-Weitzenböck formula for bundle-valued 1-form [11]

$$\{d_A, d_A^*\}\phi_\mu = \nabla_A^* \nabla_A \phi_\mu - \mathbb{R}_{\mu\bar{\nu}}\phi^{\bar{\nu}} - [(F_A)_{\mu\bar{\nu}}, \phi^{\bar{\nu}}],$$

we obtain

$$\langle F_A, [\phi, \phi^*] \rangle = \|\nabla_A \phi\|^2 - \|d_A \phi\|^2 - \|d_A^* \phi\|^2 - \langle \mathbb{R} \lrcorner \phi, \phi \rangle.$$

Therefore we arrive at

$$\mathcal{F}(A, \phi) = \|F_A\|^2 + \|[\phi, \phi^*]\|^2 + 2\|\nabla_A \phi\|^2 - 2\|d_A' \phi\|^2 - 2\|(d_A')^* \phi\|^2 + 2\|d_A'' \phi\|^2 - 2\langle \mathbb{R} \lrcorner \phi, \phi \rangle.$$

It follows from the Kähler identities $i[\Lambda, d_A'] = -(d_A'')^*$, $i[\Lambda, d_A''] = (d_A')^*$ that

$$\begin{aligned}
\|d_A' \phi\|^2 - \|d_A'' \phi\|^2 &= -i\langle \phi, [\Lambda, d_A''] d_A' \phi + [\Lambda, d_A'] d_A'' \phi \rangle \\
&= -i\langle \phi, [\Lambda F_A, \phi] - d_A' [\Lambda, d_A''] \phi \rangle \\
&= -i\langle \phi, [\Lambda F_A, \phi] \rangle - \|(d_A')^* \phi\|^2.
\end{aligned}$$

Thus we find that the functional $\mathcal{F}(A, \phi)$ can be rewritten as

$$\mathcal{F}(A, \phi) = \|F_A\|^2 + \|[\phi, \phi^*]\|^2 + 2\|\nabla_A \phi\|^2 - 2\langle \mathbb{R} \lrcorner \phi + i[\Lambda F_A, \phi], \phi \rangle.$$

For the Hermitian-Yang-Mills-Higgs metric, $i\Lambda F_A = \lambda Id - i\Lambda[\phi, \phi^*]$, thereby

$$\begin{aligned}
i\langle [\Lambda F_A, \phi], \phi \rangle &= -i\langle [\Lambda[\phi, \phi^*], \phi], \phi \rangle = i\langle [\phi, \Lambda[\phi, \phi^*]], \phi \rangle \\
&= \int_X \sum_i h([\phi^{\bar{\mu}}, \phi_\mu^*](e_i), [\phi_\nu^*, \phi^{\bar{\nu}}](e_i)) = - \int_X \sum_i h([\phi^{\bar{\mu}}, \phi_\mu^*], \phi_\nu](e_i), \phi^{\bar{\nu}}(e_i)) \\
&= - \int_X \sum_i h([\phi_\nu, \phi_\mu^*], \phi^{\bar{\mu}}](e_i), \phi^{\bar{\nu}}(e_i)) = - \int_X \sum_i h([\phi_\nu, \phi_\mu^*](e_i), [\phi^{\bar{\nu}}, (\phi^*)^\mu](e_i)) \\
&= -\|[\phi, \phi^*]\|^2.
\end{aligned}$$

Then by the semi-positivity of $\mathcal{F}(A, \phi)$, we deduce the inequality (2.3). \square

2.2. From Hitchin Pairs to Strong Yang-Mills-Higgs Pairs. In this section, we study some properties of the pairs associated with a Yang-Mills-Higgs system.

Proposition 2.2. *Let $\{(E, \phi), h, d_A\}$ be a Yang-Mills-Higgs system over an n -dimensional compact Kähler manifold (X, ω) , (A, ϕ) be the associated Hitchin pair and $0 = E_0 \subset E_1 \subset \dots \subset E_{l-1} \subset E_l = E$ be the unique Harder-Narasimhan filtration of Higgs bundle (E, ϕ) . Then the slope $\mu_\omega(E_1)$ of E_1 is not greater than the integral $\frac{\int_X \lambda(x)\omega^n}{2\pi n}$ for the largest eigenvalue $\lambda(x)$ of the Hermitian matrix $\mathcal{K}_{(A, \phi)}|_x$ at $x \in X$.*

Proof. Let us denote by E_i^\perp the orthogonal complement of E_i in E_{i+1} , then the relation between the component $\mathcal{R}_{(A, \phi)}|_{E_1}$ of $\mathcal{R}_{(A, \phi)}$ restricted on E_1 and the curvature $\mathcal{R}_{(A, \phi)}(E_1)$ with respect to the induced Hitchin-Simpson connection on E_1 is given by

$$\begin{aligned}
\mathcal{R}_{(A, \phi)}|_{E_1} - \mathcal{R}_{(A, \phi)}(E_1) &= -\alpha_1 \wedge \alpha_1^* - \alpha_2^{(1)} \wedge (\alpha_2^{(1)})^* - \dots - \alpha_{l-1}^{(l-2)} \wedge (\alpha_{l-1}^{(l-2)})^* \\
&\quad + \beta_1 \wedge \beta_1^* + \beta_2^{(1)} \wedge (\beta_2^{(1)})^* + \dots + \beta_{l-1}^{(l-2)} \wedge (\beta_{l-1}^{(l-2)})^*,
\end{aligned}$$

where $\alpha_i^{(i-1)} = \alpha_i \circ p_i \circ \cdots \circ p_2$ for $\alpha_i \in \Omega^{0,1} \otimes \text{Hom}(E_i^\perp, E_i)$ coming from the decomposition of the holomorphic structure and the natural projection p_i from E_i to E_{i-1} , $\beta_i^{(i-1)} = \beta_i \circ p_i \circ \cdots \circ p_2$ for $\beta_i \in \Omega^{1,0} \otimes \text{Hom}(E_i^\perp, E_i)$ engendered by the decomposition of the Higgs field. Then

$$\begin{aligned} \deg_\omega(E_1) &= \int_X c_1(E_1) \wedge \omega^{n-1} = \frac{i}{2\pi n} \int_X \text{Tr}(\Lambda \mathcal{R}_{(A,\phi)}(E_1)) \omega^n \\ &= \frac{1}{2\pi n} \int_X (\text{Tr}(i\Lambda \Theta|_{E_1}) - |\alpha_1|^2 - |\alpha_2^{(1)}|^2 - \cdots - |\alpha_{l-1}^{(l-2)}|^2 - |\beta_1|^2 - |\beta_2^{(1)}|^2 - \cdots - |\beta_{l-1}^{(l-2)}|^2) \omega^n \\ &\leq \frac{1}{2\pi n} \text{rank}(E_1) \int_X \lambda \omega^n, \end{aligned}$$

where the property of Hermitian matrix that any diagonal element is not bigger than the largest eigenvalue plays a crucial role. Indeed, for any Hermitian matrix $H = [H_{ij}] = U \text{diag}\{\lambda_1, \dots, \lambda_r\} U^*$ with some unitary matrix U and eigenvalues $\lambda_1, \dots, \lambda_r$, one has $H_{ii} = \sum_j \lambda_j U_{ij} \bar{U}_{ij} \leq \lambda_{\max} \sum_j U_{ij} \bar{U}_{ij} = \lambda_{\max} (= \max\{\lambda_1, \dots, \lambda_r\})$. \square

Proposition 2.3. *Let (A, ϕ) be a Yang-Mills-Higgs pair. The following facts are obvious:*

- (1) *If $[\phi, \Lambda \Theta]$ is $\Delta_{d'_A}$ -harmonic, then (A, ϕ) is a strong Yang-Mills-Higgs pair.*
- (2) *If (A, ϕ) is non-degenerate, then $\det(\Lambda \Theta)$ is a non-zero constant.*

Proposition 2.4. *Let $\{(E, \phi), h, d_A\}$ and $\{(E, \phi), \tilde{h}, d_{\tilde{A}}\}$ be two Yang-Mills-Higgs system with h, \tilde{h} being conformal to each other pointwise. If the corresponding Hitchin pairs (A, ϕ) and (\tilde{A}, ϕ) are both Yang-Mills-Higgs pairs, then \tilde{h} is just a rescaling of h .*

Proof. We write $\tilde{h} = fh$ with the conformal factor f being a positive smooth function on X . Then

$$\begin{aligned} i\Lambda \tilde{\Theta} &= \Lambda((d'_A + (\tilde{h})^{-1} d'_A \tilde{h} + d''_A)^2 + [\phi, \phi^*_{\tilde{h}}]) \\ &= i\Lambda(\Theta + \bar{\partial}(f^{-1} \partial f) Id) \\ &= i\Lambda \Theta + \left(\frac{1}{2} \Delta \ln f\right) Id, \end{aligned}$$

which shows that the right hand sides of (1.4) are conformal invariant. Thereby the exact form $d \ln f$ is harmonic, so it has to vanish, thus f is a constant. \square

Proposition 2.5. *A Hitchin pair (A, ϕ) is a strong Yang-Mills-Higgs pair if and only if it satisfies the equation*

$$\mathcal{D}_{(A,\phi)}^*(F_A + [\phi, \phi^*] + d'_A \phi + d''_A \phi^*) = 0. \quad (2.4)$$

Proof. By manipulations of the Kähler identities with Higgs field (1.1), we obtain

$$\begin{aligned} &\mathcal{D}_{(A,\phi)}^*(F_A + [\phi, \phi^*] + d'_A \phi + d''_A \phi^*) \\ &= i([\Lambda, \mathcal{D}_{(A,\phi)}'' - [\Lambda, \mathcal{D}_{(A,\phi)}'])(\Theta + d'_A \phi + d''_A \phi^*) \\ &= i(\mathcal{D}_{(A,\phi)}' - \mathcal{D}_{(A,\phi)}'')(\Lambda \Theta) + i\Lambda(\mathcal{D}_{(A,\phi)}'' - \mathcal{D}_{(A,\phi)}')(\Theta + d'_A \phi + d''_A \phi^*) \end{aligned}$$

From the Banchi identity for A , we have

$$\begin{aligned} &(\mathcal{D}_{(A,\phi)}'' - \mathcal{D}_{(A,\phi)}')(\Theta + d'_A \phi + d''_A \phi^*) \\ &= [\phi - \phi^*, \Theta] + (d''_A - d'_A)[\phi, \phi^*] + [\phi - \phi^*, d'_A \phi + d''_A \phi^*] + d''_A d'_A \phi - d'_A d''_A \phi^* \\ &= [\phi, F_A] - [\phi^*, F_A] + [F_A, \phi] - [F_A, \phi^*] \\ &= 0, \end{aligned}$$

where the second equality is due to $[\phi, d'_A \phi] = -d'_A(\phi \wedge \phi) = 0$ and the Jacobi identity which implies $[\phi, [\phi, \phi^*]] = [\phi \wedge \phi, \phi^*] = 0$. Therefore,

$$\begin{aligned} &\mathcal{D}_{(A,\phi)}^*(F_A + [\phi, \phi^*] + d'_A \phi + d''_A \phi^*) \\ &= i(d'_A - d''_A)(\Lambda \Theta) - i[\phi, \Lambda \Theta] + i[\phi^*, \Lambda \Theta]. \end{aligned}$$

As a result, the equations (2.4) reduce to

$$\begin{aligned} d'_A(\Lambda\Theta) &= [\phi, \Lambda\Theta], \\ d''_A(\Lambda\Theta) &= [\phi^*, \Lambda\Theta]. \end{aligned} \quad (2.5)$$

We have to show that both sides of the equalities in (3.6) vanish. Indeed, by Kähler identities we calculate

$$\begin{aligned} \|d''_A(\Lambda\Theta)\|^2 &= \langle d''_A(\Lambda\Theta), [\phi^*, \Lambda\Theta] \rangle = \langle \Lambda\Theta, (d'_A)^*([\phi^*, \Lambda\Theta]) \rangle \\ &= \langle \Lambda\Theta, -i[\Lambda, d'_A]([\phi^*, \Lambda\Theta]) \rangle = i\langle \Lambda\Theta, \Lambda([\phi^*, d'_A\Lambda\Theta]) \rangle \\ &= i\langle \Lambda\Theta, \Lambda([\phi^*, [\phi, \Lambda\Theta]]) \rangle \\ &= i\langle \Lambda\Theta, \Lambda([\phi, [\Lambda\Theta, \phi^*]]) \rangle - i\langle \Lambda\Theta, [\Lambda\Theta, \Lambda([\phi, \phi^*])] \rangle \\ &= i\langle \Lambda\Theta, [\Lambda, d''_A]([\phi, \Lambda\Theta]) \rangle \\ &= -\langle d'_A(\Lambda\Theta), [\phi, \Lambda\Theta] \rangle = -\|[\phi, \Lambda\Theta]\|^2 = 0, \end{aligned}$$

where the Jacobi identity has been applied to the forth equality, and the anti-Hermiticity of Θ yields $\langle \Lambda\Theta, [\Lambda\Theta, \Lambda([\phi, \phi^*])] \rangle = -\text{Re}\langle [\Lambda\Theta, \Lambda\Theta], \Lambda([\phi, \phi^*]) \rangle = 0$. This completes the proof. \square

Definition 2.6. A global holomorphic section s of the Higgs bundle (E, ϕ) is called (ϕ, κ) -invariant if for the holomorphic 1-form κ the equality $\phi(s) = \kappa \otimes s$ holds.

Proposition 2.7. Let (A, ϕ) be a strong Yang-Mills-Higgs pair associated with a Yang-Mills-Higgs system $\{(E, \phi), h, d_A\}$. If the Higgs field ϕ admits a non-trivial (ϕ, κ) -invariant section s , then we have the inequality

$$\|\widetilde{d'_A s} + \bar{\kappa} \tilde{s}\|^2 \geq \|\kappa \tilde{s}\|^2,$$

where $\tilde{s} := \Lambda\Theta(s)$. Obviously, when s is covariant constant, the equality holds.

Proof. It is known that the Kähler identities imply that

$$\Delta_{d'_A} - \Delta_{d''_A} = i[\Lambda, F_A]. \quad (2.6)$$

Therefore for any global holomorphic section s we have

$$\|d'_A s\|^2 = \langle i\Lambda F_A s, s \rangle \geq 0. \quad (2.7)$$

Moreover if s is (ϕ, κ) -invariant, then $\langle i\Lambda\Theta s, s \rangle \geq 0$ since $i\Lambda\Theta(s) = i\Lambda F_A(s)$. Now (A, ϕ) is a strong Yang-Mills-Higgs pair then $\Lambda\Theta(s)$ is also a (ϕ, κ) -invariant section since $d''_A(\Lambda\Theta) = [\phi, \Lambda\Theta] = 0$. By the Kähler identities with Higgs field we have an analog of (2.6)

$$\Delta_{\mathcal{D}'_{(A, \phi)}} - \Delta_{\mathcal{D}''_{(A, \phi)}} = i[\Lambda, \mathcal{R}_{(A, \phi)}]. \quad (2.8)$$

Applying (2.7) and (2.8) to $\Lambda\Theta(s)$ yields

$$\|\mathcal{D}'_{(A, \phi)} \Lambda\Theta(s)\|^2 \geq \|\mathcal{D}''_{(A, \phi)} \Lambda\Theta(s)\|^2.$$

This is exactly the inequality in the proposition. \square

Proposition 2.8. If a Yang-Mills-Higgs system $\{(E, \phi), h, d_A\}$ over an n -dimensional Kähler manifold (X, ω) admits a strong degenerate Yang-Mills-Higgs pair (A, ϕ) , then (A, ϕ) must be a Hermitian-Yang-Mills-Higgs pair if (E, ϕ) is a semistable Higgs bundle.

Proof. Define $F := \Lambda\Theta(E)$ that is a proper holomorphic subbundle of E since $\Lambda\Theta$ is degenerate and covariantly constant. The commutativity of $\Lambda\Theta$ and ϕ guarantees that F is a Higgs subbundle. Similarly $K := \ker(\Lambda\Theta)$ is also a non-trivial Higgs subbundle of E . There is an orthogonal decomposition of E as a C^∞ -bundle with respect to the Hermitian metric h : $E = F \oplus F^\perp$. However, we have an isomorphism of C^∞ -bundles: $F^\perp \simeq K$. In fact, if $u \in K$, then for any $v = \Lambda\Theta(s) \in F$ we have $h(u, v) = -h(\Lambda\Theta(u), s) = 0$, i.e. $K \subset F^\perp$, conversely, $F^\perp \subset K$ is also due to the anti-Hermiticity of Θ . So this C^∞ -decomposition is actually a holomorphic decomposition, which means that the second fundamental forms of the subbundles F and K vanish. Hence $R_K = \pi_K R_E \pi_K$ where R_E and R_K are

curvatures corresponding to connections d_A on E and $\pi_K d_A \pi_K$ on K respectively, and π_K stands for the projection to K . Therefore, we have

$$\begin{aligned} \deg_\omega(K) &= \frac{i}{2\pi n} \int_X \text{Tr}(\Lambda R_K) \omega^n \\ &= -\frac{i}{2\pi n} \int_X \text{Tr}_K(\Lambda[\phi, \phi^*]) \omega^n = -\frac{i}{2\pi} \int_X \text{Tr}_K([\phi, \phi^*]) \wedge \omega^{n-1} = 0, \end{aligned}$$

which implies $\deg_\omega(F) = \deg_\omega(E) - \deg_\omega(K) = \deg_\omega(E)$, thus $\mu_\omega(F) \geq \mu_\omega(E)$. This shows that $E = K$ if (E, ϕ) is a semistable Higgs bundle, then (A, ϕ) is a Hermitian-Yang-Mills-Higgs pair. \square

Corollary 2.9. *Suppose a Yang-Mills-Higgs system $\{(E, \phi), h, d_A\}$ over an n -dimensional Kähler manifold (X, ω) admits a strong degenerate Yang-Mills-Higgs pair (A, ϕ) .*

- (1) *Assume E is of rank 2, if $\deg_\omega(E) = 0$, then (A, ϕ) is a Hermitian-Yang-Mills-Higgs pair, and if $\deg_\omega(E) \neq 0$, the Harder-Narasimhan filtration associated with (E, ϕ) is exactly $0 \subset F \subset E$.*
- (2) *If (E, ϕ) is a semistable Higgs bundle over $X = \mathbb{P}^1$, then E decomposes orthogonally into the direct sum of trivial Higgs line bundles.*
- (3) *Assume the rank of E is not less than 2, and X is an elliptic curve, then (E, ϕ) cannot be a stable Higgs bundle.*

Proof. (1) Suppose (A, ϕ) is not a Hermitian-Yang-Mills-Higgs pair, then F is a line bundle such that $i\Lambda\Theta|_F = \kappa$. $\Lambda\Theta$ being covariantly yields that κ is a non-zero constant which shows $\deg_\omega(E) = \deg_\omega(F) \neq 0$.

(2) Since $\deg(E) = 0$, according to the classical Grothendieck theorem, we have $E \simeq \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_r)$, where the sum of integers m_1, \dots, m_r that are unique up to the permutation is zero. It follows from the Higgs-semistability of E that if not all $m_i (i = 1, \dots, r)$ vanish then for some positive integer n there is a non-trivial morphism $\tilde{\phi} : \mathcal{O}(k) \rightarrow \mathcal{O}(k' - 2)$ ($n' \neq n$) that factors through the Higgs morphism $\phi|_{\mathcal{O}(k)} : \mathcal{O}(k) \rightarrow E \otimes \mathcal{O}(-2)$ and the natural projection. Therefore there is a non-zero element belongs to $H^0(\mathbb{P}^1, \mathcal{O}(k' - k - 2))$ which means that $k' \geq k + 2 > k$. Hence, the semi-stability and zero degree together guarantee that there is no component as $\mathcal{O}(k), k \neq 0$. As a result, $m_1 = \cdots = m_r = 0$, then the Higgs field on the trivial bundle $E \simeq \mathcal{O}^{\oplus r}$ should to be also trivial. Fix a component $L \simeq \mathcal{O}$ of E so that $E \simeq L \oplus L^\perp$, then $F_A|_L = \bar{\partial}(f^{-1}\partial f) - \alpha \wedge \alpha^*$ where f is a smooth function, and $\alpha \in \Omega^{0,1} \otimes \text{Hom}(L^\perp, L)$ denotes the second fundamental form. Hence $i\Lambda\Theta|_L = \frac{1}{2}\Delta \ln f + |\alpha|^2 = 0$. By Hopf's maximum principle, f has to be a constant, thus α must vanish, in other words, $L^\perp \simeq \mathcal{O}^{\oplus r-1}$ as the holomorphic bundles. Then the induction on the rank r of E gives the conclusion.

(3) Suppose (E, ϕ) is Higgs-stable, thus (E, h, ϕ) is a harmonic Higgs bundle. Firstly, we claim that E has to split. Indeed, if not so, by Atiyah's results[12], E is isomorphic to $E' \otimes L$ where E' is an indecomposable bundle of rank r and degree zero with a global section, and L is a line bundle of degree zero, moreover, there is an exact sequence $0 \rightarrow \mathcal{O}_\Sigma \rightarrow E' \rightarrow E'' \rightarrow 0$ for an indecomposable bundle E'' of rank $r - 1$ and degree zero, which implies L is a proper subbundle of E . Since the canonical line bundle of Σ is trivial, the Higgs field ϕ induces a morphism $\tilde{\phi} : L \rightarrow E$, or a morphism $\tilde{\phi} : \mathcal{O}_X \rightarrow E'$. By the Higgs-stability of E , the composition $p \circ \tilde{\phi} \in H^0(X, \mathcal{O}_X)$ of $\tilde{\phi}$ and the projection $p : E \rightarrow L$ cannot be an isomorphism, which exhibits a contradiction. Hence we deduce the claim. Now from the decomposition $E = E_1 \oplus E_2$ one easily sees that E is not stable. Assume E is strictly semistable, then $\deg_\omega(E_1) = \deg_\omega(E_2) = 0$. By recursion, we find that E can be decomposed into the direct sum of indecomposable holomorphic bundles of degree zero, i.e. $E = \oplus E_i$. Let $\tilde{\phi}_i = p_i \circ \phi_i$ be the composition of $\phi_i = \phi|_{E_i} : E_i \rightarrow E$ and the projection $p_i : E \rightarrow E_i$, then $(E_i, \tilde{\phi}_i)$ are all Higgs bundles. By the same arguments, for each E_i , if $\tilde{\phi}_i \neq 0$ there is a line subbundle $L_i \subseteq E_i$ of degree zero such that the restriction of $\tilde{\phi}_i$ on L_i is an isomorphism, namely L_i is a ϕ -invariant proper subbundle of E , which will contradict with the Higgs-stability of E . The remaining case is $\tilde{\phi}_i = 0$ for $\forall i$, then for each E_i there is $E_j, j \neq i$, such that $\Phi_{ij} = p_j \circ \phi_i : E_i \rightarrow E_j$ is a non-zero morphism. Write $E_i = L_i \otimes E'_i$, and it follows from the fact that $(E'_i)^\vee \simeq E'_i$ and the multiplicative structure[12] of E'_i 's that $\Phi_{ij}|_{L_i} : L_i \xrightarrow{\sim} L_j$ which again contradicts with the Higgs-stability. So far, we only need to prove that if E is not semistable, then

(E, ϕ) is not Higgs semistable. To show it, we consider the Harder-Narasimhan filtration of E : $0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E$, and let E_i be the smallest subbundle among them containing $\phi(E_1)$. If $\phi(E_1) \neq 0$, we get a non-zero morphism $\bar{\phi} : E_1 \rightarrow \text{Gr}_i(E)$ by the composition of ϕ and taking quotient. But it is known that $H^0(X, \text{Hom}(E_1, \text{Gr}_i(E))) = 0$ if $i > 1$ since E_1 and $\text{Gr}_i(E)$ are all semistable and $\mu_\omega(E_1) > \mu_\omega(\text{Gr}_i(E))$. Thus E_1 is a Higgs subbundle with $\mu_\omega(E_1) > \mu_\omega(E)$. We complete the proof. \square

Definition 2.10. ([13]) Let (E, ϕ) be a Higgs bundle over a Kähler manifold X . We call the following complex of coherent \mathcal{O}_X -modules:

$$\mathcal{E}^\bullet = (E \xrightarrow{\phi} E \otimes \Omega_X^1 \xrightarrow{\phi} E \otimes \Omega_X^2 \rightarrow \cdots)$$

the Higgs complex, and define the Higgs cohomology $H^i(X, (E, \phi)) := \mathbb{H}^i(X, \mathcal{E}^\bullet)$ to be the hypercohomology of the Higgs complex.

Corollary 2.11. Suppose a Yang-Mills-Higgs system $\{(E, \phi), h, d_A\}$ admits a strong degenerate Yang-Mills-Higgs pair (A, ϕ) .

- (1) If (E, ϕ) is a non-Higgs-semistable bundle of rank 2 and $X = \mathbb{P}^n$ ($n \geq 2$), then the Higgs cohomologies $H^i(\mathbb{P}^n, (E(m), \tilde{\phi}))$ vanish for all $m \in \mathbb{Z}$, $i = 1, \dots, n-1$, where $(E(k) := E \otimes \mathcal{O}_{\mathbb{P}^n}(k), \tilde{\phi} := \phi|_E \otimes \text{Id}|_{\mathcal{O}_{\mathbb{P}^n}(k)})$ is regarded as a Higgs bundle.
- (2) If (E, ϕ) is Higgs-stable with $\text{rank}(E) \geq 2$, then the Higgs cohomology $H^0(X, (E, \phi))$ vanishes, and if X is an n -dimensional Calabi-Yau manifold, the Higgs cohomology $H^{2n}(X, (E, \phi))$ is also vanishes.

Proof. (1) Since $\Lambda\Theta \neq 0$, (E, ϕ) splits into the direct sum of two Higgs line bundles, thus $(E, \phi) = (F, \phi|_F) \oplus (K, \phi|_K)$, where $F \simeq \mathcal{O}_{\mathbb{P}^n}(k)$ for $k = \deg(E)$, $K \simeq \mathcal{O}_{\mathbb{P}^n}$. However, we note that $\phi|_F, \phi|_K \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1)$ should be zero because $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1) = H^{1,0}(\mathbb{P}^n, \mathbb{C}) = \overline{H^{0,1}(\mathbb{P}^n, \mathbb{C})} = \overline{H^1(X, \mathcal{O}_{\mathbb{P}^n})} = 0$ when $n \geq 2$, thereby $H^i(\mathbb{P}^n, (E(k), \tilde{\phi})) = H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k+m)) \oplus H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $i = 1, \dots, n-1$.

(2) The complex of sheaves of C^∞ -sections $\mathcal{A}^0(E) \xrightarrow{\mathcal{D}''_{(A, \phi)}} \mathcal{A}^1(E) \xrightarrow{\mathcal{D}''_{(A, \phi)}} \mathcal{A}^2(E) \rightarrow \cdots$ gives a fine resolution of the Higgs complex. Therefore the hypercohomology of the Higgs complex is isomorphic to the cohomology of the complex of global sections $\Gamma(\mathcal{A}^0(E)) \xrightarrow{\mathcal{D}''_{(A, \phi)}} \Gamma(\mathcal{A}^1(E)) \xrightarrow{\mathcal{D}''_{(A, \phi)}} \Gamma(\mathcal{A}^2(E)) \rightarrow \cdots$. Assume that there is a non-trivial section $s \in \Gamma(\mathcal{A}^0(E))$ satisfies $d_A'' s = \phi(s) = 0$, thus ϕ can be viewed as a $(\phi, 0)$ -invariant section. Then since $\Lambda\Theta = 0$, we have $d_A'' s = 0$, which means s may generate a flat line bundle $L \subset E$ with the trivial Higgs field. Hence $\deg(L) = \deg(E) = 0$ that will contradict with the Higgs-stability of E . For a Calabi-Yau manifold X , $\Omega_X^n \simeq \mathcal{O}_X$, then by Serre duality, $\mathbb{H}^{2n}(X, \mathcal{E}^\bullet) \simeq (\mathbb{H}^0(X, (\mathcal{E}^\bullet)^\vee))^\vee$. The Hitchin pair on the dual stable Higgs bundle $(E^\vee, h^\vee, \phi^\vee)$ is also a strong degenerate Yang-Mills-Higgs pair, so the previous conclusion implies the vanishing of $\mathbb{H}^{2n}(X, \mathcal{E}^\bullet)$. \square

Let $\{(E, \phi), h, d_A\}$ be the Yang-Mills-Higgs system as that in the Proposition 2.8. If the limit $\lim_{c \rightarrow 0} (E, c\phi)$ exists (for example, when X is a smooth projective manifold), it must be a fixed point of the \mathbb{C}^* -action which implies that it carries the structure of a system of Hodge bundles. More precisely, this means that E with respect to the limiting holomorphic structure splits holomorphically as a sum $E = \bigoplus_{i=1}^l E_i$ and that the limiting Higgs field is given by a collection of holomorphic maps $\phi_i : E_i \rightarrow E_{i+1} \otimes \Omega_X^1$, $1 \leq i \leq k$ (with the convention that $E_{l+1} = 0$). Moreover each subbundle E_i splits as $E_i = F_i \oplus K_i$ with $F_i \subset F$, $K_i \subset K$ and ϕ_i decomposes as $\phi_i = \theta_i \oplus \vartheta_i$ with $\theta_i : F_i \rightarrow F_{i+1} \otimes \Omega_X^1$, $\vartheta_i : K_i \rightarrow K_{i+1} \otimes \Omega_X^1$, thus $(\bigoplus_{i=1}^l F_i, \bigoplus_{i=1}^l \theta_i)$ and $(\bigoplus_{i=1}^l K_i, \bigoplus_{i=1}^l \vartheta_i)$ are both systems of Hodge bundles. Since the deformation changes the holomorphic structures of E, F and K , but not their isomorphism classes as differentiable complex vector bundles, hence their degrees remain unchanged, namely $\deg_\omega(\bigoplus F_i) = \deg_\omega(E)$ and $\deg_\omega(\bigoplus K_i) = 0$.

Corollary 2.12. Suppose X is a smooth projective manifold. If $c_2(E) = c_2(F)$, the holomorphic tangent bundle T_X is a semistable bundle and the map ϑ_{l-1} is a non-zero injective map, then $\mu_\omega(K_{l-1}) \leq$

$\frac{n}{n+1}\mu_\omega(\Omega_X^1)$. In particular, if X is a Riemann surface and ϑ_{l-1} satisfies the same assumption as above, we have $\mu_\omega(K_{l-1}) \leq \frac{1}{2}\mu_\omega(\Omega_X^1)$.

Proof. Since $\Lambda\Theta|_K = 0$, thus $(K, \phi|_K)$ is a polystable Higgs bundle with trivial first Chern class, $c_2(K) = c_2(E) - c_2(F) = 0$, then K is a harmonic Higgs bundle, thereby the corresponding limiting systems of Hodge bundles is a variation of Hodge structures which implies that $(\oplus_{i=1}^l K_i, \oplus_{i=1}^l \vartheta_i)$ is a semistable Higgs bundle. Let P be the first term in the Harder-Narasimhan filtration of K_{l-1} , which is semistable since P has maximal slope among the subbundles of K_{l-1} . Consider the injective map $\vartheta_{l-1} : K_{l-1} \otimes T_X^1 \rightarrow K_l$, then $P \oplus \vartheta_{l-1}(P \otimes T_X)$ is a Higgs subbundle due to the vanishing of ϑ_l . Therefore $\deg_\omega(P) + \deg_\omega(P \otimes T_X) \leq 0$. Since $P \otimes T_X$ is a semistable bundle, then we have

$$\mu_\omega(P \otimes T_X) = \mu_\omega(P) + \mu_\omega(T_X) \leq -\frac{\deg_\omega(P)}{\text{rank}(P \otimes T_X)},$$

thus

$$(\text{rank}(P \otimes T_X) + \text{rank}(P))\mu_\omega(P) \leq \text{rank}(P \otimes T_X)\mu_\omega(\Omega_X^1).$$

Hence

$$\mu_\omega(K_{l-1}) \leq \mu_\omega(P) \leq \frac{\text{rank}(P \otimes T_X)}{\text{rank}(P \otimes T_X) + \text{rk}(P)}\mu_\omega(\Omega_X^1) = \frac{n}{n+1}\mu_\omega(\Omega_X^1).$$

□

3. STABLE YANG-MILLS-HIGGS PAIRS

3.1. Deformation of the Hitchin Pair. Let $(A_t = A_0 + t\alpha, \phi_t = \phi_0 + t\beta)$ be a family of the Hitchin pairs with one parameter $t \in \mathbb{C}$, where (A_0, ϕ_0) is a fixed Hitchin pair, $\alpha \in \mathcal{A}_X^{1,0}(\text{End}(E))$, $\beta \in \mathcal{A}_X^{1,0}(\text{End}(E))$. Then the deformation pair $(\alpha^{0,1}, \beta)$ is subject to the following equations

$$\begin{aligned} td_{A_0}''\alpha^{0,1} + t^2\alpha^{0,1} \wedge \alpha^{0,1} &= 0, \\ td_{A_0}''\beta + t[\alpha^{0,1}, \phi_0] + t^2[\alpha^{0,1}, \beta] &= 0, \\ t[\phi_0, \beta] + t^2\beta \wedge \beta &= 0. \end{aligned} \tag{3.1}$$

Definition 3.1. (1) If $d_{A_0}''\alpha^{1,0} = d_{A_0}''\beta = 0$, then $(\alpha^{0,1}, \beta)$ is called the holomorphic deformation pair.

(2) If one expresses $\alpha = \alpha_0 + \sum_{i \geq 1} \alpha_i t^i$ and $\beta = \beta_0 + \sum_{i \geq 1} \beta_i t^i$, then $(a_0^{0,1}, \beta_0)$ is called the infinitesimal deformation pair.

The infinitesimal deformation pair $(a_0^{0,1}, \beta_0)$ satisfies $\mathcal{D}_{(A_0, \phi_0)}''(a_0^{0,1} + \beta_0) = 0$, thus $[a_0^{0,1} + \beta_0] \in H^1(X, (\text{End}(E), \tilde{\phi}_0))$ where $(\text{End}(E), \tilde{\phi}_0)$ is viewed as a Higgs bundle via the induced Higgs field $\tilde{\phi}_0 = \phi_0|_E \otimes \text{Id}|_{E^\vee} + \text{Id}|_E \otimes \phi_0^\vee|_{E^\vee}$. Let $\pi : \Omega_X^1 \rightarrow X$ be the holomorphic cotangent bundle on X , then we have $E \simeq \pi_*\mathcal{S}$ and $(E \xrightarrow{\phi_0} E \otimes \Omega_X^1) \simeq \pi_*(\mathcal{S} \otimes (\mathcal{O}_{\Omega_X^1} \xrightarrow{\Phi} \pi^*\Omega_X^1))$ for a locally free $\mathcal{O}_{\Omega_X^1}$ -sheaf \mathcal{S} where $\Phi \in H^0(\pi^*\Omega_X^1)$ is the tautological section[5], which induces a Koszul complex[14]

$$K_\bullet(\Phi) = (0 \rightarrow \wedge^n \pi^*T_X \rightarrow \wedge^{n-1} \pi^*T_X \rightarrow \cdots \rightarrow \pi^*T_X \xrightarrow{\Phi^\vee} \mathcal{O}_{\Omega_X^1} \rightarrow 0).$$

The zero scheme of Φ can be identified with X , hence $K_\bullet(\Phi)$ is a projective resolution of \mathcal{O}_X . By definition, $\text{Ext}^i(\mathcal{O}_X \otimes_{\mathcal{O}_{\Omega_X^1}} \mathcal{S}, \mathcal{S}) = \mathbb{H}^i(\text{Hom}(K_\bullet(\Phi) \otimes_{\mathcal{O}_{\Omega_X^1}} \mathcal{S}, \mathcal{S})) = \mathbb{H}^i(K_\bullet^\vee(\Phi) \otimes_{\mathcal{O}_{\Omega_X^1}} \mathcal{E}nd(\mathcal{S})) = \mathbb{H}^i(\pi_*(K_\bullet^\vee(\Phi) \otimes_{\mathcal{O}_{\Omega_X^1}} \mathcal{E}nd(\mathcal{S})))$, therefore $[a_0^{0,1} + \beta_0] \in \text{Ext}^1(\mathcal{O}_X \otimes_{\mathcal{O}_{\Omega_X^1}} \mathcal{S}, \mathcal{S}) = \text{Ext}^1(\mathcal{O}_X, \mathcal{E}nd_{\mathcal{O}_{\Omega_X^1}}(\mathcal{S}))$.

Proposition 3.2. If $\text{Ext}^2(\mathcal{O}_X, \mathcal{E}nd_{\mathcal{O}_{\Omega_X^1}}(\mathcal{S})) = 0$, the solution of equations (3.1) exists.

Proof. From (3.1) the equations that the higher order terms of α, β should obey read

$$\mathcal{D}_{(A_0, \phi_0)}''(a_k^{0,1} + \beta_k) + \sum_{i+j=k-1} (a_i^{0,1} + \beta_j) \wedge (a_j^{0,1} + \beta_j) = 0 \tag{3.2}$$

for $\forall k \geq 1$. Let \mathcal{H}^k denote the space of harmonic k -forms valued in $\text{End}(E)$ corresponding to the Laplacian $\Delta_{\mathcal{D}_{(A_0, \phi_0)}''}$, then there are isomorphisms $H^k(X, \text{End}(E)) \simeq \mathbb{H}^k(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \simeq \mathcal{H}^k$, and we have

the operator $Q^{(k)} : L^2(\mathcal{A}_X^k(\text{End}(E))) \rightarrow L^2(\mathcal{A}_X^k(\text{End}(E)))$ such that $\Delta_{\mathcal{D}''_{(A_0, \phi_0)}} \circ Q^{(k)} = Id - p_{\mathcal{H}^k}$ where $p_{\mathcal{H}^k}$ stands for the projection on the space \mathcal{H}^k . One can easily check that $Q^{(k)}$ commutes with $\mathcal{D}''_{(A_0, \phi_0)}$ and $(\mathcal{D}''_{(A_0, \phi_0)})^*$, then define the operator $G^{(k)} = (\mathcal{D}''_{(A_0, \phi_0)})^* \circ Q^{(k)} = Q^{(k)} \circ (\mathcal{D}''_{(A_0, \phi_0)})^* : L^2(\mathcal{A}_X^k(\text{End}(E))) \rightarrow L^2(\mathcal{A}_X^{k-1}(\text{End}(E)))$.

By assumption, $\Delta_{\mathcal{D}''_{(A_0, \phi_0)}} \circ Q^{(2)} = Id$, hence $\{\mathcal{D}''_{(A_0, \phi_0)}, G^{(2)}\} = Id$. We put

$$\begin{aligned}\alpha_k^{0,1} &= -p^{0,1} \left(\sum_{i+j=k-1} G^{(2)}((a_i^{0,1} + \beta_j) \wedge (a_j^{0,1} + \beta_j)) \right) \\ \beta_k &= -p^{1,0} \left(\sum_{i+j=k-1} G^{(2)}((a_i^{0,1} + \beta_j) \wedge (a_j^{0,1} + \beta_j)) \right),\end{aligned}$$

where p denotes the projection on the corresponding space, which provide the desired solution of (3.2) via the induction on k .

In order to complete the proof, we have to show the formal power series $\alpha^{0,1} = \alpha_0^{0,1} + \sum_{k \geq 1} \alpha_k^{0,1} t^k$ and $\beta = \beta_0 + \sum_{k \geq 1} \beta_k t^k$ are convergent, and furthermore, are smooth sections. The method applied here is standard due to Kodaira-Spencer[15]. We denote by $\|\xi\|_s$ the Sobolev norm of the section $\xi \in \mathcal{A}_X^k(\text{End}(E))$ which is given by the sum of the L^2 -norms of i -th derivative of ξ for all $i \leq s$, where s is a sufficiently large integer compared to $2 \dim_{\mathbb{C}} X$. It follows from the standard estimate of elliptic differential operators that

$$\begin{aligned}\|G^{(2)}((a_i^{0,1} + \beta_j) \wedge (a_j^{0,1} + \beta_j))\|_s &< C_1 \|((a_i^{0,1} + \beta_j) \wedge (a_j^{0,1} + \beta_j))\|_{s-1} \\ &< C_2 \|a_i^{0,1} + \beta_j\|_s \|a_j^{0,1} + \beta_j\|_s\end{aligned}$$

with positive constants C_1, C_2 depend only on s and the manifold X . Then by induction on k there exists a constant C_3 such that

$$\|a_k^{0,1} + \beta_k\|_s < \left[\frac{k+1}{2}\right] C_3^k \|a_0^{0,1} + \beta_0\|_s^{k+1}.$$

Therefore, if we choose suitably $a_0^{0,1} + \beta_0 \in \mathcal{H}^1$ such that $|t| \|a_0^{0,1} + \beta_0\|_s$ is sufficiently small, then we may deduce the convergence. By Sobolev's fundamental lemma, $a^{0,1} + \beta \in C^m(\mathcal{A}_X^1(\text{End}(E)))$ for $m = s-1 - \dim_{\mathbb{C}} X$. On the other hand, we note that $\Delta_{\mathcal{D}''_{(A_0, \phi_0)}}(a^{0,1} + \beta) + (\mathcal{D}''_{(A_0, \phi_0)})^*((a^{0,1} + \beta) \wedge (a^{0,1} + \beta)) = 0$ which is an elliptic PDE for sufficiently small $a^{0,1} + \beta$, hence $a^{0,1} + \beta \in C^\infty(\mathcal{A}_X^1(\text{End}(E)))$. \square

3.2. Stable Yang-Mills-Higgs Pairs. Let us consider the second variation of the Yang-Mills-Higgs functional, thus we calculate

$$\frac{d^2}{dt^2} \Big|_{t=0} YMH(A_t, \phi_t) = \text{Re} \langle \Upsilon, \mathcal{D}_{(A_0, \phi_0)}^* \mathcal{D}_{(A_0, \phi_0)} \Upsilon + \Upsilon^* \lrcorner \mathcal{R}_{(A_0, \phi_0)} \rangle,$$

where $\Upsilon = -(\alpha^{0,1})^* + \alpha^{0,1} + \beta + \beta^* \in \mathcal{A}_X^1(\text{End}(E))$ associated with the deformation pair $(\alpha^{0,1}, \beta)$.

Definition 3.3. A strong Yang-Mills-Higgs pair (A_0, ϕ_0) on a holomorphic vector bundle (E, h) is called the semi-stable (stable, unstable) Yang-Mills-Higgs pair along the given deformation pair $(\alpha^{0,1}, \beta)$ if the following condition is satisfied

$$\text{Re} \langle \Upsilon, \mathcal{D}_{(A_0, \phi_0)}^* \mathcal{D}_{(A_0, \phi_0)} \Upsilon + \Upsilon^* \lrcorner \mathcal{R}_{(A_0, \phi_0)} \rangle \geq (>, <) 0, \quad (3.3)$$

and is called the weakly semi-stable Yang-Mills-Higgs pair if for the arbitrary admitted holomorphic deformation pair $(\alpha^{0,1} \neq 0, \beta)$ the inequality (3.3) holds.

Proposition 3.4. If (A_0, ϕ_0) is a stable Yang-Mills-Higgs pair along the deformation pair $(\alpha^{0,1}, \beta)$, then we have the inequality

$$Q_{(A_0, \phi_0)}(\alpha, \tilde{\beta}) := i \langle \alpha \circ \alpha + \tilde{\beta} \circ \tilde{\beta}, \Lambda \Theta_{(A_0, \phi_0)} \rangle + \|d_{A_0}^* \alpha\|^2 - \|\tilde{\phi}_0 \circ \tilde{\beta}\|^2 > 0, \quad (3.4)$$

where $Q_{(A_0, \phi_0)}$ is a hermitian quadratic form on the space of deformation pairs, $\alpha = -(\alpha^{0,1})^* + \alpha^{0,1}$, $\tilde{\beta} = \beta + \beta^*$, $\tilde{\phi}_0 = \phi_0 + \phi_0^*$, and the action \circ is defined by

$$\Omega \circ \Xi = \Omega \lrcorner \Xi^{1,0} + \Xi^{0,1} \lrcorner \Omega$$

for any $\Omega \in \mathcal{A}^k(\text{End}(E))$ and $\Xi \in \mathcal{A}^1(\text{End}(E))$. In particular, if α is parallel with respect to the Hitchin-Simpson connection associated with (A_0, ϕ_0) , the inequality (3.4) is rewritten as

$$\begin{aligned} & 2i\langle \alpha, \Lambda\Theta_{(A_0, \phi_0)} \star \alpha \rangle - i\langle \alpha, \Lambda(\Theta_{(A_0, \phi_0)} \star \alpha) \rangle + \|\nabla_{A_0} \alpha\|^2 - \langle \alpha, \mathbb{R} \circ \alpha \rangle \\ & - i\langle \tilde{\beta}, \Lambda\Theta_{(A_0, \phi_0)} \star \tilde{\beta} \rangle - \|\tilde{\phi}_0 \circ \tilde{\beta}\|^2 > 0, \end{aligned} \quad (3.5)$$

where the action \star is defined by

$$\Omega \star \Xi = [\Omega, \Xi^{1,0}] + [\Xi^{0,1}, \Omega].$$

Proof. From the proof of Proposition 2.5, we have seen that

$$\begin{aligned} \frac{d}{dt}|_{t=0} YMH(A_t, \phi_t) &= 2i\langle (\alpha^{0,1})^*, d'_{A_0} \Lambda\Theta_{(A_0, \phi_0)} \rangle + 2i\langle \beta, [\phi_0, \Lambda\Theta_{(A_0, \phi_0)}] \rangle \\ &+ 2i\langle \alpha^{0,1}, d''_{A_0} \Lambda\Theta_{(A_0, \phi_0)} \rangle - 2i\langle \beta^*, [\phi_0^*, \Lambda\Theta_{(A_0, \phi_0)}] \rangle. \end{aligned}$$

Therefore, the second variation is given by

$$\begin{aligned} & \frac{d^2}{dt^2}|_{t=0} YMH(A_t, \phi_t) \\ &= -2i\langle (\alpha^{0,1})^*, [(\alpha^{0,1})^*, \Lambda\Theta_{(A_0, \phi_0)}] - id'_{A_0}(d'_{A_0})^*(\alpha^{0,1})^* - id'_{A_0}(d''_{A_0})^*\alpha^{0,1} \rangle \\ &+ 2i\langle (\alpha^{0,1})^*, d'_{A_0} \Lambda([\beta, \phi_0^*] + [\phi_0, \beta^*]) \rangle + 2i\langle \beta, [\beta, \Lambda\Theta_{(A_0, \phi_0)}] \rangle \\ &+ 2\langle \beta, [\phi_0, (d'_{A_0})^*(\alpha^{0,1})^* + (d''_{A_0})^*\alpha^{0,1}] \rangle + 2i\langle \beta, [\phi_0, \Lambda([\beta, \phi_0^*] + [\phi_0, \beta^*])] \rangle \\ &+ 2i\langle \alpha^{0,1}, [\alpha^{0,1}, \Lambda\Theta_{(A_0, \phi_0)}] + id''_{A_0}(d'_{A_0})^*(\alpha^{0,1})^* + id''_{A_0}(d''_{A_0})^*\alpha^{0,1} \rangle \\ &+ 2i\langle \alpha^{0,1}, d''_{A_0} \Lambda([\beta, \phi_0^*] + [\phi_0, \beta^*]) \rangle - 2i\langle \beta^*, [\beta^*, \Lambda\Theta_{(A_0, \phi_0)}] \rangle \\ &- 2\langle \beta^*, [\phi_0^*, (d'_{A_0})^*(\alpha^{0,1})^* + (d''_{A_0})^*\alpha^{0,1}] \rangle - 2i\langle \beta^*, [\phi_0^*, \Lambda([\beta, \phi_0^*] + [\phi_0, \beta^*])] \rangle \\ &= -4i\text{Re}\langle \alpha^{0,1} \lrcorner (\alpha^{0,1})^* + \beta \lrcorner \beta^*, \Lambda\Theta_{(A_0, \phi_0)} \rangle + 4\|(d'_{A_0})^*(\alpha^{0,1})^*\|^2 + 4\text{Re}\langle (d'_{A_0})^*(\alpha^{0,1})^*, (d''_{A_0})^*\alpha^{0,1} \rangle \\ &- 4\|\phi_0 \lrcorner \beta^*\|^2 - 4\text{Re}\langle \phi_0 \lrcorner \beta^*, \beta \lrcorner \phi_0^* \rangle \\ &= -4i\langle \alpha^{0,1} \lrcorner (\alpha^{0,1})^* + \beta \lrcorner \beta^*, \Lambda\Theta_{(A_0, \phi_0)} \rangle + 2\|d_{A_0}^* \alpha\|^2 - 2\|\phi_0 \lrcorner \beta^* + \beta \lrcorner \phi_0^*\|^2, \end{aligned}$$

which exhibits the Proposition the inequality (3.4). If $\mathcal{D}_{(A_0, \phi_0)} \alpha = 0$, then α is d_{A_0} -closed and $\tilde{\phi}_0$ -invariant. By means of the Bochner-Weitzenböck formula which leads to

$$\begin{aligned} \|d_{A_0}^* \alpha\|^2 &= \|\nabla_{A_0} \alpha\|^2 - \langle \alpha, \mathbb{R} \circ \alpha + \Theta_{(A_0, \phi_0)} \circ \alpha \rangle \\ &= \|\nabla_{A_0} \alpha\|^2 - \langle \alpha, \mathbb{R} \circ \alpha \rangle - i\langle \alpha, \Lambda(\Theta_{(A_0, \phi_0)} \star \alpha) - \Lambda\Theta_{(A_0, \phi_0)} \star \alpha \rangle, \end{aligned}$$

we immediately obtain the inequality the inequality (3.5). \square

Example 3.5. Assume that v is a non-zero (1,0)-type vector field that is parallel with respect to the connection determined by the Kähler metric on X , and Π is a non-zero $\Delta_{d_{A_0}}$ -harmonic (1,1)-form valued in $\text{End}(E)$, i.e. $d_{A_0} \Pi = d_{A_0}^* \Pi = 0$, and they together satisfy

$$(\nabla_{A_0})_v \Pi = 0, [\phi_0, v \lrcorner \Pi] = 0, [v \lrcorner \Pi, v \lrcorner \Pi] = 0,$$

then $v \lrcorner \Pi$ is also $\Delta_{d_{A_0}}$ -harmonic, since $d_{A_0}(v \lrcorner \Pi) = (\nabla_{A_0})_v \Pi - v \lrcorner d_{A_0} \Pi = 0$, and $\langle d_{A_0}^*(v \lrcorner \Pi), \theta \rangle = \langle \Pi, \bar{v}^\vee \wedge d_{A_0} \theta \rangle = -\langle d_{A_0}^* \Pi, \bar{v}^\vee \wedge \theta \rangle = 0$ for $\forall \theta \in C^\infty(\text{End}(E))$. Hence $(\alpha^{0,1}, \beta) = (v \lrcorner \Pi, \phi_0)$ gives rise to a holomorphic deformation. Moreover if the chosen pair (A_0, ϕ_0) is Hermitian and degenerate, then $Q(\alpha, \beta) = -4\|[\phi_0, \phi_0^*]\|^2$, thus (A_0, ϕ_0) is not stable along such deformation direction.

Corollary 3.6. (1) A Hermitian-Yang-Mills-Higgs pair (A_0, ϕ_0) is stable along the deformation pair $(\alpha^{0,1}, \beta)$ if and only if we have

$$\|d_{A_0}^* \alpha\|^2 > \|\phi_0 \lrcorner \beta^* + \beta \lrcorner \phi_0^*\|^2. \quad (3.6)$$

(2) A stable Yang-Mills-Higgs pair (A_0, ϕ_0) along the deformation pair $(\alpha^{0,1}, \beta)$ on a Riemann surface satisfies

$$\langle \tilde{\beta}, F_{A_0} \circ \tilde{\beta} - [\phi_0, \tilde{\beta}] \lrcorner \phi_0 - \phi_0^* \lrcorner [\tilde{\beta}, \phi_0^*] \rangle + \langle \alpha, d_{A_0} d_{A_0}^* \alpha - \Theta_{(A_0, \phi_0)} \circ \alpha \rangle > 0. \quad (3.7)$$

- (3) If there exists a weakly semi-stable Yang-Mills-Higgs pair (A_0, ϕ_0) with the property that $\Lambda\Theta_{(A_0, \phi_0)}$ and $\Lambda[\phi_0, \phi_0^*]$ are both non-zero on a compact Kähler manifold X , then the singular homology $H_1(X, \mathbb{R})$ vanishes.

Proof. (1) If (A_0, ϕ_0) is the Hermitian-Yang-Mills-Higgs pair the stability condition (3.3) obviously reduces to the inequality (3.6).

(2) For the case of Riemann surface, we should note that the operator Λ on 2-forms is an isometry with respect to the Kähler metric, hence

$$\begin{aligned} 4i\langle \beta \lrcorner \beta^*, \Lambda\Theta_{(A_0, \phi_0)} \rangle &= -2\langle \beta + \beta^*, \Theta_{(A_0, \phi_0)} \lrcorner \beta + \beta^* \lrcorner \Theta_{(A_0, \phi_0)} \rangle \\ &= -2\langle \beta + \beta^*, F_{A_0} \circ (\beta + \beta^*) \rangle - 4\langle [\beta, \beta^*], [\phi_0, \phi_0^*] \rangle \\ &= -2\langle \beta + \beta^*, F_{A_0} \circ (\beta + \beta^*) \rangle - 4\|[\beta, \phi_0^*]\|^2, \end{aligned}$$

where the third equality is due to the Jacobi identity. Then one can easily check the inequality (3.7).

(3) By $\partial\bar{\partial}$ -lemma, the space V consisting of the closed $(0,1)$ -forms on X is isomorphic to the cohomology $H^{0,1}(X, \mathbb{C})$. Then we take $\alpha^{0,1} = v\Lambda\Theta_{(A_0, \phi_0)}$ for an element $v \in V$, thus $\alpha = 2\text{Re}(v)\Lambda\Theta_{(A_0, \phi_0)}$, and $\beta = c\phi_0$ for a nonzero constant c . It follows that $(\alpha^{0,1}, \beta)$ forms a holomorphic deformation pair from (A_0, ϕ_0) being a strong Yang-Mills-Higgs pair. Thereby we have

$$Q_{(A_0, \phi_0)}(\alpha, \tilde{\beta}) = \langle \alpha, \Delta_{A_0} \alpha \rangle - 4|c|^2 \|[\phi_0, \phi_0^*]\|^2.$$

Since $\Lambda[\phi_0, \phi_0^*] \neq 0$ means $[\phi_0, \phi_0^*] \neq 0$ and c can be chosen to be sufficiently large such that $Q_{(A_0, \phi_0)}(\alpha, \tilde{\beta}) < 0$, $H^{0,1}(X, \mathbb{C})$ has to vanish, thus $H^1(X, \mathbb{C})$ must also vanish by Hodge decomposition theorem. \square

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